

\hbar -expansion for the Schrödinger equation with a position-dependent mass

D. A. Kulikov*

*Theoretical Physics Department, FFEKS, Dniepropetrovsk National University
72 Gagarin avenue, Dniepropetrovsk 49010, Ukraine*

V. M. Shapoval†

*Bogolyubov Institute for Theoretical Physics
14-b Metrolohichna str., Kiev 03680, Ukraine*

Abstract

A recursion technique of obtaining the asymptotical expansions for the bound-state energy eigenvalues of the radial Schrödinger equation with a position-dependent mass is presented. As an example of the application we calculate the energy eigenvalues for the Coulomb potential in the presence of position-dependent mass and we derive the inequalities regulating the shifts of the energy levels from their constant-mass positions.

Keywords: position-dependent mass, bound state, energy spectrum, \hbar -expansion.

PACS number(s): 03.65.-w, 03.65.Ge, 03.65.Sq

Introduction

The notion of effective mass, which was introduced more than 50 years ago to describe the motion of electrons in crystals [1], has now a broader scope. It is employed in nuclear physics [2], theory of quantum liquids [3], metallic clusters [4] and semiconductor heterostructures: quantum wells, wires and dots [5]. Attempts to accommodate spatial inhomogeneity of multi-layer nanostructures have invoked treatment of the effective mass as a position-dependent quantity [6, 7, 8].

In order to calculate the bound-state spectrum of the Schrödinger equation with a position-dependent mass (PDM), various iterative [9, 10], variational [11, 12] and perturbative [13] schemes were proposed. Usually, these schemes provide solutions in numerical form. On the other hand, there exist methods of deriving exact analytical solutions to the PDM Schrödinger equation for some potentials such as the Coulomb,

*kulikov_d_a@yahoo.com, kulikov@dsu.dp.ua

†shapoval@bitp.kiev.ua

oscillator and Morse potentials [14, 15, 16]. Having a solution in analytical form greatly simplifies investigation of the structure of energy levels. Thus an important task is the development of approximate analytical methods to solve the bound-state problem for the PDM Schrödinger equation.

For the constant-mass Schrödinger equation with spherical symmetry, the $1/N$ -expansion method has proved to be efficient. The name of $1/N$ -expansion refers to the group of related approaches [17, 18, 19, 20, 21] that start with describing classical motion of a particle located at the bottom of the effective-potential well and then consider quantum corrections. The versions of this method differ in choosing a formal expansion parameter. Namely, going to the classical limit, one takes for the small parameter the inverse number of spatial dimensions, the inverse principal quantum number, or the Planck constant. The last choice, the so-called \hbar -expansion [20], seems to be the most natural one and, besides, allows to formalize calculations in a simple recursion procedure.

The purpose of this work is to extend the \hbar -expansion technique to the PDM Schrödinger equation. We develop the procedure for calculating the bound-state energy eigenvalues and then apply it to investigate the influence of PDM on the Coulomb potential spectrum.

1 \hbar -expansion technique

The PDM Hamiltonian of the particle is given by [22]

$$\hat{H} = \hat{T} + V(\mathbf{r}), \quad \hat{T} = -\frac{\hbar^2}{4} [m(\mathbf{r})^\alpha \nabla m(\mathbf{r})^\beta \nabla m(\mathbf{r})^\gamma + m(\mathbf{r})^\gamma \nabla m(\mathbf{r})^\beta \nabla m(\mathbf{r})^\alpha], \quad (1)$$

where $m(\mathbf{r})$ is the PDM, $V(\mathbf{r})$ is the interaction potential, α, β, γ are the ambiguity parameters ($\alpha + \beta + \gamma = -1$).

Let us consider the spherically symmetric case in which the stationary Schrödinger equation with the Hamiltonian (1) is reduced to the radial equation

$$\left\{ \frac{d^2}{dr^2} + \frac{m'(r)}{m(r)} \left(\frac{1}{r} - \frac{d}{dr} \right) - \frac{l(l+1)}{r^2} - \frac{2m(r)}{\hbar^2} (U(r) - E) \right\} \psi(r) = 0,$$

$$U(r) = V(r) - \frac{\hbar^2}{2} \left[\frac{\alpha + \gamma}{2} \left(\frac{m''(r)}{m^2(r)} + \frac{2m'(r)}{m^2(r)r} \right) - (\alpha\gamma + \alpha + \gamma) \frac{m'^2(r)}{m^3(r)} \right], \quad (2)$$

with l being the orbital quantum number and primes denoting derivatives with respect to $r = |\mathbf{r}|$.

Assuming $V(r)$ and $m(r)$ are analytical functions such that Eq. (2) possesses bound states, we intend to find the discrete energy spectrum. To that end, we generalize the \hbar -expansion technique [20] to the PDM case.

First we recast Eq. (2) into the Riccati equation for the logarithmic derivative of the wave function. Upon substituting $\psi(r) = \chi(r)\sqrt{m(r)}$ to eliminate the term with $d\psi(r)/dr$ and then putting $C(r) = \hbar\chi'(r)/\chi(r)$, we get

$$\hbar C'(r) + C^2(r) = \frac{\hbar^2 l(l+1)}{r^2} + \left(\frac{3}{4} + \alpha\gamma + \alpha + \gamma \right) \hbar^2 Q^2(r) - \frac{\hbar^2}{r} Q(r) -$$

$$-\frac{1 + \alpha + \gamma}{2} \hbar^2 P(r) + 2m(r)(V(r) - E) \quad (3)$$

where we denoted $Q(r) = m'(r)/m(r)$, $P(r) = (m''(r) + 2m'(r)/r)/m(r)$.

We try to obtain the solution to the Riccati equation with series expansions in the Planck constant \hbar

$$C(r) = \sum_{k=0}^{\infty} C_k(r) \hbar^k, \quad E = \sum_{k=0}^{\infty} E_k \hbar^k. \quad (4)$$

In view of the freedom in defining the classical limit, which is formally the limit of $\hbar \rightarrow 0$, the centrifugal contribution can be rewritten as follows

$$\hbar^2 l(l+1) = \Lambda^2 + \hbar A \Lambda + \hbar^2 B \quad (5)$$

that enables to consider different modifications of the method depending on values of parameters A and B [20]. A particular choice of these values amounts to fixing a zeroth approximation. In practice, one first specifies $V(r)$ and $m(r)$ and then choose A and B in such a way that higher-order corrections in the \hbar -expansions for eigenenergies (4) become as small as possible. Note that one may achieve yet faster convergence of this series expansion by means of the renormalization: rewriting $V(r)$ or $m(r)$ as the \hbar -expansion and properly adjusting its coefficients. Such generalizations of the method lie however out of the scope of the present article.

Substituting the expansions (4) into the Riccati equation (3) and equating to zero coefficients of successive powers of \hbar , we obtain the system of equations

$$\begin{aligned} C_0^2 &= 2m(r)(V(r) - E_0) + \frac{\Lambda^2}{r^2}, \\ C_0' + 2C_0C_1 &= -2m(r)E_1 + \gamma_1 \left(\frac{r_0}{r}\right)^2, \\ C_1' + 2C_0C_2 + C_1^2 &= -2m(r)E_2 + \gamma_2 \left(\frac{r_0}{r}\right)^2 + F(r), \\ &\dots \\ C_{k-1}' + \sum_{i=0}^k C_i C_{k-i} &= -2m(r)E_k + \gamma_k \left(\frac{r_0}{r}\right)^2, \quad k > 2. \end{aligned} \quad (6)$$

Here $\gamma_1 = A\Lambda/r_0^2$, $\gamma_2 = B/r_0^2$, $\gamma_3 = \gamma_4 = \dots = 0$,

$$F(r) = \left(\frac{3}{4} + \alpha\gamma + \alpha + \gamma\right) Q^2(r) - \frac{1}{r}Q(r) - \frac{1 + \alpha + \gamma}{2}P(r). \quad (7)$$

In order to take into account the nodes of the wave function for radially excited states, we employ the Zwaan-Dunham quantization condition [23, 24] which expresses the principle of argument in complex analysis. For the discrete spectrum, solutions to Eq. (2) are real on the real axis and possess a finite number of simple nodes on it. The number of these nodes is just the radial quantum number n_r . Then the integration of the logarithmic derivative $C(r)$ along the contour that encloses only the above nodes yields

$$\frac{1}{2\pi i} \oint C(r) dr = n_r \hbar, \quad n_r = 0, 1, 2, \dots \quad (8)$$

This condition must be supplemented with a rule of achieving the classical limit $\hbar \rightarrow 0$ for the radial and orbital quantum numbers which are the specific quantum notions. We choose this rule as follows

$$\hbar \rightarrow 0, \quad n_r = \text{const}, \quad l \rightarrow \infty, \quad \hbar n_r \rightarrow 0, \quad \hbar l = \text{const}. \quad (9)$$

Physically, it means that in the classical limit the particle is located in the minimum of the effective potential $V(r) + \Lambda^2/(2m(r)r^2)$ and thus moves along the stable circular orbit whose radius r_0 is a solution to the equation

$$m(r_0)r_0^3V'(r_0) = \Lambda^2 \left(1 + \frac{m'(r_0)r_0}{2m(r_0)} \right). \quad (10)$$

Meanwhile, the energy in the zeroth approximation is given by

$$E_0 = V(r_0) + \frac{\Lambda^2}{2m(r_0)r_0^2} \quad (11)$$

and the quantization rule transforms into

$$\frac{1}{2\pi i} \oint C_k(r) dr = n_r \delta_{k,1}. \quad (12)$$

With the help of such rules we can develop an algebraic recursion procedure for calculation of the \hbar -expansion corrections to energies avoiding bulky formulas that arise in higher orders of the standard Rayleigh-Schrödinger perturbation theory.

2 Recursion procedure for calculating energies

To solve Eqs. (6), we expand the functions $V(r)$, $m(r)$, $F(r)$, $P(r)$ and $Q(r)$ into Taylor series in powers of the new variable, the deviation from the minimum of the effective potential, $x = (r - r_0)/r_0$,

$$\begin{aligned} V(r) &= \sum_{i=0}^{\infty} V_i x^i, & m(r) &= \sum_{i=0}^{\infty} m_i x^i, & F(r) &= \sum_{i=0}^{\infty} F_i x^i, \\ P(r) &= \sum_{i=0}^{\infty} P_i x^i, & Q(r) &= \sum_{i=0}^{\infty} Q_i x^i \end{aligned} \quad (13)$$

where $V_i = r_0^i V^{(i)}(r_0)/i!$, $m_i = r_0^i m^{(i)}(r_0)/i!$, whereas F_i , P_i and Q_i are expressed through V_i , m_i .

Consider the first equation of Eqs. (6) that determines $C_0(r)$. Writing down the squared quantity as

$$C_0^2(r) = \omega^2 x^2 (1 + a_1 x + a_2 x^2 + \dots) \quad (14)$$

with

$$\begin{aligned} \omega &= \sqrt{2(m_0 V_2 + m_1 V_1) + \frac{2m_0 V_1}{2m_0 + m_1} (3m_0 - m_2)}, \\ a_i &= \frac{2}{\omega^2} \left(\sum_{j=0}^{i+1} m_j V_{i-j+2} + \frac{m_0^2 V_1}{2m_0 + m_1} \left((i+3)(-1)^i - \frac{m_{i+2}}{m_0} \right) \right), \end{aligned} \quad (15)$$

we find the coefficients of the Taylor series expansion for $C_0(x)$

$$C_0(x) = -\omega x \sqrt{1 + a_1 x + a_2 x^2 + \dots} = x \sum_{i=0}^{\infty} C_i^0 x^i, \quad (16)$$

$$C_0^0 = -\omega, \quad C_i^0 = \frac{1}{2\omega} \left(\sum_{j=1}^{i-1} C_j^0 C_{i-j}^0 - \omega^2 a_i \right). \quad (17)$$

The "−" sign in Eq. (16) is chosen so as to meet the boundary conditions: $C_0(r) > 0$ for $r < r_0$, $C_0(r) < 0$ for $r > r_0$, which follow from the quadratic integrability of the wave function.

The function $C_0(x)$ has a simple zero at $x = 0$. Then, after examining the system of equations (6), we conclude that $C_k(x)$ has a pole of the order of $(2k - 1)$ at this point and, hence, can be represented by the Laurent series

$$C_k(x) = x^{1-2k} \sum_{i=0}^{\infty} C_i^k x^i. \quad (18)$$

With definition of residues, this expansion permits us to express the quantization conditions (12) explicitly in terms of the coefficients C_i^k

$$C_{2k-2}^k = \frac{n_r}{r_0} \delta_{1,k}. \quad (19)$$

Inserting the above expansions into Eqs. (6) and collecting coefficients of powers of x , we obtain recursion formulas for calculating the yet undetermined C_i^k and also the corrections to eigenenergies E_k

$$\begin{aligned} C_i^k = \frac{1}{2C_0^0} & \left[\Theta(2 - 2k + i) \left(\gamma_k (-1)^i (3 - 2k + i) + F_{i-2} \delta_{k,2} - \right. \right. \\ & \left. \left. - \frac{m_{i+2-2k}}{m_0} \left(\gamma_k - \frac{1}{r_0} C_{2k-2}^{k-1} - \sum_{j=0}^k \sum_{p=0}^{2k-2} C_p^j C_{2k-2-p}^{k-j} \right) \right) - \frac{3 - 2k + i}{r_0} C_i^{k-1} - \right. \\ & \left. - \sum_{j=1}^{k-1} \sum_{p=0}^i C_p^j C_{i-p}^{k-j} - 2 \sum_{p=1}^i C_p^0 C_{i-p}^k \right], \quad i \neq 2k - 2, \end{aligned} \quad (20)$$

$$E_k = \frac{1}{2m_0} \left(\gamma_k + F_0 \delta_{k,2} - \frac{1}{r_0} C_{2k-2}^{k-1} - \sum_{j=0}^k \sum_{p=0}^{2k-2} C_p^j C_{2k-2-p}^{k-j} \right) \quad (21)$$

where $\Theta(k)$ is the Heaviside step-like function ($\Theta(k) = 1$ for $k \geq 0$).

The derived formulas completely resolve the task of calculating the energy spectrum of the PDM Schrödinger equation.

As an illustration, let us compute the energy eigenvalues in the field of the Coulomb potential $V(r) = -q/r$ in the presence of the PDM

$$m(r) = \frac{m_c}{(1 + ar)^\lambda} \quad (22)$$

where the constants q , m_c and a are assumed to be positive. We choose the parameters Λ , A and B in the form

$$\Lambda = \hbar(n_r + l + 1), \quad A = -2n_r - 1, \quad B = n_r^2 + n_r, \quad (23)$$

which in the constant-mass case guarantees that the Balmer formula for energies is reproduced just in the zeroth approximation E_0 of the \hbar -expansion method. The ambiguity parameters are set to be $\alpha = \gamma = 0$, $\beta = -1$ in correspondence with the kinetic part of the effective Hamiltonian obtained for a crystal with slowly-varying inhomogeneity [6, 25].

The calculated partial sums $E^{(k)} = E_0 + E_1\hbar + \dots + E_k\hbar^k$ of the \hbar -expansions for energies as well as the exact energy values E_{num} found by numerical integration are listed for various λ in Table 1. The calculation has been carried out for the states with $n_r = 0$, $l = 2$ and $n_r = 1$, $l = 1$, using $q = 10$, $a = 0.1$ in units in which $\hbar = 2m_c = 1$.

Table 1: Absolute values of partial sums $E^{(k)}$ of \hbar -expansions for energies in the Coulomb potential in the presence of the PDM $m(r) = m_c/(1 + ar)^\lambda$. E_{num} is the result of numerical integration.

k	$(n_r = 0, l = 2)$				$(n_r = 1, l = 1)$			
	$\lambda = 2$	$\lambda = 3$	$\lambda = -2$	$\lambda = -3$	$\lambda = 2$	$\lambda = 3$	$\lambda = -2$	$\lambda = -3$
	$ E^{(k)} $							
0	1.77778	1.18817	3.64395	4.04566	1.77778	1.18817	3.64395	4.04566
1	1.94444	1.45345	3.50153	3.83753	2.27778	1.98401	3.21669	3.42127
2	1.83333	1.25876	3.58047	3.94991	2.07556	1.66974	3.39891	3.69143
3	1.83000	1.24978	3.57934	3.94732	2.05556	1.61180	3.39064	3.67228
4	1.83111	1.25190	3.58015	3.94898	2.06000	1.62647	3.39352	3.67887
5	1.83111	1.25184	3.58014	3.94896	2.06000	1.62478	3.39330	3.67837
	$ E_{\text{num}} $							
	1.83111	1.25183	3.58014	3.94897	2.06000	1.62510	3.39329	3.67834

It should be added that in the particular case of $\lambda = 2$ the problem has an exact solution [15] and the \hbar -expansion technique restores it. Indeed, in this case we have $E_5 = E_6 = \dots = 0$ and $E^{(4)}$ coincides with the exact value, as seen from Table 1.

3 Order of energy levels

Let us apply the obtained recursion formulas to determine the order of energy levels in the Coulomb field in the presence of PDM.

First of all, we notice that for the chosen values of the ambiguity parameters $\alpha = \gamma = 0$, $\beta = -1$ there exists an inequality making comparison of the energy levels with their constant-mass positions. Namely, if $m(r)$ and m_c are the PDM and the constant-mass respectively, the corresponding energy eigenvalues $E(n_r, l)$ and $E_c(n_r, l)$ satisfy

$$E(n_r, l) \geq E_c(n_r, l) \quad \text{if} \quad \forall r \in [0, \infty) \quad m(r) \leq m_c, \quad (24)$$

provided that these eigenvalues exist.

The inequality (24) holds valid not only for the Coulomb, but also for an arbitrary attractive potential since it is a direct consequence of the variational definition of the discrete spectrum of a Hamiltonian bounded below [26]. In particular, this inequality explains the increase of binding energy for impurity bound states in quantum dots that was predicted in [27].

To obtain further inequalities, we fix the shape of the PDM (22) and consider the first and second \hbar -expansion corrections whose contribution to the energy is decisive. According to Eqs. (11) and (21), we have

$$\begin{aligned} E_0 &= -\frac{q}{r_0} + \frac{\Lambda^2}{2m_c r_0^2} (1 + ar_0)^\lambda, \\ E_1 &= \frac{(2n_r + 1)(1 + ar_0)^\lambda (\omega r_0 - \Lambda)}{2m_c r_0^2} \end{aligned} \quad (25)$$

where

$$\omega = \sqrt{\frac{m_c q (1 + ar_0)^{-\lambda-1} [a^2 r_0^2 (2 - \lambda)(1 - \lambda) + 2ar_0(2 - \lambda) + 2]}{r_0(2 + (2 - \lambda)ar_0)}}. \quad (26)$$

It is convenient to rewrite the sum of these two corrections in the form

$$E_0 + \hbar E_1 = E_B + \frac{m_c q^2}{2s^2 \Lambda^2} \left[t^\lambda - 1 + (s - 1)^2 + \frac{\hbar(2n_r + 1)(\omega r_0/\Lambda - 1)}{\Lambda} t^\lambda \right] \quad (27)$$

with $E_B = -m_c q^2 / 2\hbar^2 (n_r + l + 1)^2$ being the constant-mass energy, $t = 1 + ar_0$, $s = t^\lambda - \frac{\lambda}{2}(t^\lambda - t^{\lambda-1})$.

Using the expression (27), we can establish the relative order of energy levels that refer to the same value of the principal quantum number $n = n_r + l + 1$ and are degenerate in the constant-mass case. The quantities Λ , r_0 , ω , t and s , appeared in Eq. (27), depend on n , but not on n_r and l separately, so they cannot remove the degeneracy. It is however removed by the last term in the braces due to the factor $(2n_r + 1)$. The sign of this term coincides with that of the quantity $(\omega r_0/\Lambda - 1)$ for which we find

$$\frac{\omega r_0}{\Lambda} - 1 = \frac{\omega^2 r_0^2 / \Lambda^2 - 1}{\omega r_0 / \Lambda + 1} = -\frac{\lambda ar_0 [2 + (3 - \lambda)ar_0]}{2(\omega r_0 / \Lambda + 1)(1 + ar_0)^2}. \quad (28)$$

Obviously, if $\lambda < 0$, then $(\omega r_0/\Lambda - 1) > 0$ and, as a consequence, the level with the quantum numbers (n_r, l) has the higher energy than the level with $(n_r - 1, l + 1)$.

Now let us prove that for $\lambda > 0$ the relative order of those levels is converse, i.e., $E(n_r, l) < E(n_r - 1, l + 1)$. As seen from Eq. (28), it is sufficient to demonstrate that one has $2 + (3 - \lambda)ar_0 > 0$ when $\lambda > 0$. Rewriting the equation (10) for r_0 as

$$r_0 = \frac{\Lambda^2}{m_c q} \left[t^\lambda - \frac{\lambda}{2}(t^\lambda - t^{\lambda-1}) \right] = \frac{\Lambda^2}{m_c q} s, \quad (29)$$

we see that it has a positive root only if $s > 0$. This entails $ar_0 = t - 1 < 2/(\lambda - 2)$ for $\lambda > 0$ and, hence, $2 + (3 - \lambda)ar_0 > ar_0 > 0$ that completes the proof.

Thus the inclusion of the PDM $m(r) = m_c/(1 + ar)^\lambda$ removes the "accidental" degeneracy in the Coulomb spectrum so that

$$E(n_r, l) \leq E(n_r - 1, l + 1) \quad \text{if} \quad \lambda \geq 0. \quad (30)$$

Interestingly, in the case of $\lambda < 0$ the above order of the energy levels coincides with that observed in muonic atoms and also in the soft-core Coulomb potential [28].

To proceed, let us consider spacings between the levels with the same value of $n = n_r + l + 1$ and evaluate the ratio

$$R = \frac{E(n_r - 1, l + 1) - E(n_r, l)}{E(n_r, l) - E(n_r + 1, l - 1)}. \quad (31)$$

We adopt the expression for the energy up to the third correction, i.e., $E = E_0 + E_1\hbar + E_2\hbar^2$. Here E_2 calculated according to Eq. (21) reads

$$\begin{aligned} E_2 = & \frac{t^{\lambda-2}}{16m_c\omega^2r_0^4}(n_r^2 + n_r) \left[-8\Lambda^2(ar_0(\lambda - 2) - 2)^2 - 8\Lambda\omega r_0(6(1 + a_1) - \right. \\ & 3a^2r_0^2(\lambda - 2)(1 + \lambda + a_1) + ar_0(12(1 + a_1) - \lambda(3a_1 - 4))) - \\ & \omega^2r_0^2(a^2r_0^2(16\lambda^2 + 15a_1^2 + 8\lambda(3a_1 + 1) - 12a_2 - 8) + 2ar_0(15a_1^2 + 12\lambda a_1 - \\ & \left. 12a_2 - 8) + 15a_1^2 - 12a_2 - 8) \right] + f(n) \equiv \frac{t^{\lambda-2}}{16m_c\omega^2r_0^4}(n_r^2 + n_r)g + f(n) \end{aligned} \quad (32)$$

where $f(n)$ is a function that depends on n , but not on n_r and l separately. Its explicit form is inessential for us since we consider the levels with the same n .

Combining Eqs. (27) and (32), we find the requisite ratio

$$R = \frac{1 + (2n_r - 1)b_1/b_2}{1 + (2n_r + 1)b_1/b_2} \quad (33)$$

where

$$b_1 = \frac{t^{\lambda-2}\hbar^2g}{16m_c\omega^2r_0^4}, \quad (34)$$

$$b_2 = \left[\frac{m_cq^2t^\lambda\hbar}{\Lambda^3s^2} \left(\frac{\omega r_0}{\Lambda} - 1 \right) + \frac{t^{\lambda-2}\hbar^2g}{16m_c\omega^2r_0^4} \right]. \quad (35)$$

Clearly, one has $R < 1$ for $b_1/b_2 > 0$, $R > 1$ for $b_1/b_2 < 0$. We managed to deduce the sign of b_1/b_2 analytically in the case of the slowly-varying PDM. Assuming a is small and performing expansions in its powers, we get $b_1/b_2 = \hbar/(\hbar - 2\Lambda) + O(a)$. Then from Eq. (23) it follows that $b_1/b_2 < 0$.

Thus we conjecture the inequality

$$\frac{E(n_r, l + 1) - E(n_r, l)}{E(n_r, l) - E(n_r + 1, l - 1)} > 1. \quad (36)$$

Its correctness was checked by an exact calculation of the energy via numerical integration for various, not exclusively small, values of a and λ . This inequality says that the shift on the energy scale is greater for levels with larger orbital quantum number.

It is worthy to add that a similar problem, the PDM Dirac equation with the Coulomb potential, was considered in [29, 30]. There it was shown that relativistic factors such as an additional spin-orbit coupling due to the PDM also affect the order of energy levels.

Conclusion

In the work we have developed the technique for the approximate calculation of the energy eigenvalues of the radial PDM Schrödinger equation. Based on the expansions in powers of the Planck constant with the subsequent analysis of the system near the effective-potential minimum, this technique provides, in principle, the calculation of the energy up to an arbitrary order in the analytical or numerical form.

As an example, the spectrum of the Coulomb potential in the presence of the PDM $m(r) = m_c/(1+ar)^\lambda$ has been investigated and the inequalities regulating the order of the energy levels in this system have been established. In particular, the inequalities (30) and (36) show how the "accidental" degeneracy of the Coulomb spectrum is removed due to the PDM.

The expressions for energies obtained with the developed technique can also be applied to study the dependence of the spectra of spherically symmetric nanostructures on their radiuses that is an actual problem of modern nanophysics.

References

- [1] J. C. Slater, Phys. Rev. **76**, 1592 (1949); H. M. James, Phys. Rev. **76**, 1611 (1949); J. M. Luttinger and W. Kohn, Phys. Rev. **97**, 869 (1955).
- [2] B. D. Serot, J. D. Walecka, Adv. Nucl. Phys. **16**, 1 (1986).
- [3] F. Arias de Saavedra, J. Boronat, A. Polls, A. Fabrocini, Phys. Rev. B **50**, 4248 (1994).
- [4] A. Puente, L. Serra, M. Casas, Z. Phys. D **31**, 283 (1995).
- [5] G. Bastard, *Wave Mechanics Applied to Semiconductor Heterostructures* (Editions de Physique, Les Ulis, 1988).
- [6] K. Young, Phys. Rev. B **39**, 13434 (1989).
- [7] G. L. Herling, M. L. Rustgi, J. Appl. Phys. **71**, 796 (1992).
- [8] E. Borovitskaya, M. S. Shur, Solid-State Electronics **44**, 1609 (2000).
- [9] Y. Li, O. Voskoboynikov, C. P. Lee, S. M. Sze, Solid State Commun. **120**, 79 (2001).
- [10] H. Voss, J. Comput. Phys. **217**, 824 (2006).
- [11] E. A. Muljarov, E. A. Zhukov, V. S. Dneprovskii, Y. Masumoto, Phys. Rev. B **62**, 7420 (2000).
- [12] M. Tkach, O. Makhanets, A. Hryshchuk, J. Phys. Stud. **11**, 220 (2007).
- [13] M. Tkach, V. Holovatsky, O. Makhanets, M. Dovganiuk, J. Phys. Stud. **13**, 4706 (2009).
- [14] A. D. Alhaidari, Phys. Rev. A **66**, 042116 (2002).
- [15] C. Quesne, V. M. Tkachuk, J. Phys. A: Math. Gen. **37**, 4267 (2004).

- [16] B. Bagchi, A. Banerjee, C. Quesne, V. M. Tkachuk, J. Phys. A: Math. Gen. **38**, 2929 (2005).
- [17] L. Mlodinow, M. Shatz, J. Math. Phys. **25**, 943 (1984).
- [18] T. Imbo, A. Pagnamenta, U. Sukhatme, Phys. Rev. D **29**, 1669 (1984).
- [19] V. M. Vainberg, V. D. Mur, V. S. Popov, A. V. Sergeev, A. V. Shcheblykin, Teor. Mat. Fiz. **74**, 399 (1988).
- [20] S. S. Stepanov, R. S. Tutik, Teor. Mat. Fiz. **90**, 208 (1992).
- [21] I. O. Vakarchuk, J. Phys. Stud. **6**, 46 (2002).
- [22] O. von Roos, Phys. Rev. B **27**, 7547 (1983).
- [23] A. Zwaan, *Intensitäten in Ca-Funkenspectrum*: Thesis (Utrecht, 1929).
- [24] J. L. Dunham, Phys. Rev. **41**, 713 (1932).
- [25] D. J. BenDaniel, C. B. Duke, Phys. Rev. **152**, 683 (1966).
- [26] M. Reed, B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators* (Academic, New York, 1978).
- [27] A. John Peter, K. Navaneethakrishnan, Physica E **40**, 2747 (2008).
- [28] R. L. Hall, N. Saad, K. D. Sen, H. Ciftci, Phys. Rev. A **80**, 032507 (2009).
- [29] I. O. Vakarchuk, J. Phys. A: Math. Gen. **38**, 4727 (2005).
- [30] I. O. Vakarchuk, J. Phys. A: Math. Gen. **38**, 7567 (2005).